**Infinite Series and Comparison Tests**

Of all the tests you have seen do far and will see later, these are the trickiest to use because you have to have some idea of what it is you are trying to prove. If a series is divergent and you erroneously believe it is convergent, then applying these tests will lead only to extreme frustration. But there are some pointers that you can use in order to apply them successfully.

Here are the two tests we will study:

---

**The Basic Comparison Test**

Suppose $a_n \geq 0$ for all $n \geq m$, $m$ some nonnegative integer. Then:

- If $a_n \leq c_n$ and $\sum c_n$ converges, so does $\sum a_n$
- If $d_n \leq a_n$ and $\sum d_n$ diverges, so does $\sum a_n$

---

This is intuitively clear; let us look at the convergence case where we have a series $\sum a_n$ which satisfies $a_n \leq c_n$ and where we know that $\sum c_n$ converges. Then the situation with the partial sums, which from a monotonic increasing sequence, is this:

- $s_1 \leq s'_1$
- $s_2 \leq s'_1$
- $s_1 \leq s'_2$
- $\ldots$
- $\ldots$
- $s_n \leq s'_n$

Where $s_n$ refers to the partial sums of the series $\sum a_n$ and $s'_n$ to that of $\sum c_n$. Being this the case, it follows (not immediately obvious but intuitive) that:
Thus, if \( \lim_{n \to \infty} s_n \leq \lim_{n \to \infty} s'_n \),

\[
\sum_{n=1}^{\infty} c_n \text{ converges, } \lim_{n \to \infty} s'_n \text{ exists and so must } \lim_{n \to \infty} s_n.
\]

A similar argument can be used for the divergence case.

In order to apply this test to a given series \( \sum a_n \), we must have a feeling as to whether it converges or diverges. If we feel it converges, then we need to come up with a sequence \( \{c_n\} \) whose terms bound \( a_n \) from above \( (a_1 n \leq c_1 n) \) and whose corresponding sum \( \sum_{n=1}^{\infty} c_n \) converges.

If, on the other hand, we feel that the given series diverges, then we need to come up with a sequence \( \{d_n\} \) whose terms bound \( a_n \) from below and whose corresponding sum \( \sum_{n=1}^{\infty} d_n \) diverges.

How do we get this feeling? With practice and through the intuition that practice gives us. But there are some pointers you can follow.

Ask: what does \( a_n \) look like for large \( n \)? We will use the symbol \( a_n \sim b_n \) to mean that for large \( n \) the terms are of the same order of magnitude or behave more or less the same way. Thus, \( a_n \sim \frac{1}{n^2} \) means that \( a_n \) behaves like a \( p \)-series with \( p = 2 \) and therefore \( \sum_{n=1}^{\infty} a_n \) is expected to converge. We usually seek \( p \)-series to make comparisons because we know under what circumstances they converge or diverge.

A few examples will illustrate this concept.

**Example 1** Determine whether \( \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}} \) converges or diverges.

**Solution**

We begin by factoring the largest power of \( n \) in order to determine what \( a_n \) looks like for large \( n \):
Thus, for large $n$

\[
\frac{1}{n + \sqrt{n}} \sim \frac{1}{n} \quad \frac{1}{n \left(1 + \frac{1}{\sqrt{n}}\right)} \sim \frac{1}{n}
\]

and we expect divergence. Therefore, we seek a sequence $\{d_n\}$ with $d_n \leq \frac{1}{n + \sqrt{n}}$ and such that $\sum d_n$ is known to diverge.

$? \leq \frac{1}{n + \sqrt{n}}$

What should we put in place of “?”?

We need to make the fraction $\frac{1}{n + \sqrt{n}}$ smaller. We can achieve this by making its numerator smaller and/or its denominator larger. Not much we can do with the numerator, but we know this:

$n \geq \sqrt{n}$ for $n \geq 1$

Keep this identity in mind because you will use it substantially.

Thus,

$n \geq \sqrt{n} \Rightarrow n + n \geq n + \sqrt{n}$

or

$2n \geq n + \sqrt{n}$

From which it follows that

\[
\frac{1}{2n} \leq \frac{1}{n + \sqrt{n}}
\]

Since $\sum \frac{1}{2n}$ diverges (why?), $\sum \frac{1}{n + \sqrt{n}}$ also diverges and we are done.
Example 2 Determine whether \( \sum_{n=6}^{\infty} \frac{1}{5n^2 - n} \) converges or diverges.

Solution

Once again, factoring the largest power of \( n \) in order to determine what \( \alpha_n \) looks like for large \( n \):

\[
\frac{1}{5n^2 - n} = \frac{1}{n^2 \left(\frac{5}{n} + \frac{1}{n}\right)} \sim \frac{1}{n^2}
\]

Therefore, we expect convergence and we need to come up with a \( c_n \) such that

\[
\frac{1}{5n^2 - n} \leq c_n
\]

and such that \( \sum c_n \) is known to converge. So now we have to make our fraction larger. How do we do this? We make the numerator larger and/or the denominator smaller.

\[
\frac{1}{5n^2 - n} \leq ?
\]

Once again, not much we can do with the numerator although \( \frac{1}{5n^2 - n} \leq \frac{n}{5n^2 - n} \) is tempting, but it leads nowhere; we still don’t know what \( \frac{n}{5n^2 - n} \) does.

How do we make the denominator \( 5n^2 - n \) smaller? We take more away from \( n^2 \): how about

\[
5n^2 - n \geq 5n^2 - n^2 = 4n^2
\]

Thus,

\[
\frac{1}{5n^2 - n} \leq \frac{1}{4n^2}
\]

\[
\sum \frac{1}{4n^2}
\]

Since \( \sum \frac{1}{4n^2} \) (why?), so does \( \sum \frac{1}{5n^2 - n} \).
Example 3 Determine whether \( \sum_{n=1}^{\infty} \frac{\ln n}{n^3} \) converges or diverges.

Solution

The operational thing to observe here is that for \( n \geq \ln(n) \) which can be seen from the graph below:

Thus we can write

\[
\frac{\ln n}{n^3} \sim \frac{n}{n^3} = \frac{1}{n^2}
\]

and we expect convergence. Thus, we seek to bound \( \frac{\ln n}{n^3} \) from above and the choice is clear in light of the inequality \( n \geq \ln(n) \):

\[
\frac{\ln n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}
\]

Since \( \sum_{n=6}^{\infty} \frac{1}{n^2} \) converges, so does \( \sum_{n=6}^{\infty} \frac{\ln n}{n^3} \).

Example 4 Determine whether \( \sum_{k=1}^{\infty} \frac{k^2 \ln(k)}{k^3 + 1} \) converges or diverges.

Solution
We need to get an idea of how \( \frac{k^2 \ln(k)}{k^3 + 1} \) behaves in terms of the series which we know to converge or diverge. Using the inequality \( \ln(k) \leq k \) we can write

\[
\frac{k^2 \ln(k)}{k^3 + 1} \leq \frac{k^3}{k^3 + 1}
\]

But this does not help because although we know \( \sum_{k=1}^{\infty} \frac{k^2}{k^3 + 1} \) diverges (why?) we still don’t know what \( \sum_{k=1}^{\infty} \frac{k^2 \ln(k)}{k^3 + 1} \) does. However, we still have a feeling that \( \sum_{k=1}^{\infty} \frac{k^2 \ln(k)}{k^3 + 1} \) diverges; there is enough in the numerator to make it grow fast enough relative to its denominator. So we need to bound its terms from below with terms whose sum diverges:

\[
\sum_{k=1}^{\infty} \frac{k^2 \ln(k)}{k^3 + 1} 
\]

Since \( \ln(k) > 1 \) for \( k > e \approx 2.718 \), we can write

\[
\frac{k^2}{k^3 + 1} \leq \frac{k^2 \ln(k)}{k^3 + 1}
\]

for \( k \geq 3 \). Now we are getting somewhere. We can also write

\[
\frac{k^2}{k^3 + k^2} \leq \frac{k^2 \ln(k)}{k^3 + 1}
\]

And finally, the left-most expression reduces to \( \frac{k^2}{k^3 + k^2} = \frac{1}{k + 1} \). Thus,

\[
\sum_{k=1}^{\infty} \frac{1}{k + 1} \leq \frac{k^2 \ln(k)}{k^3 + 1}
\]

Since \( \sum_{k=1}^{\infty} \frac{1}{k + 1} \) (why?) diverges, so does \( \sum_{k=1}^{\infty} \frac{k^2 \ln(k)}{k^3 + 1} \).

Our next comparison test is a little more mechanical in nature and it is called the Limit Comparison Test:

**The Limit Comparison Test**

Suppose \( a_n \geq 0 \) and \( b_n > 0 \) be sequences and

\[
r = \lim_{n \to \infty} \frac{a_n}{b_n}
\]

Then

\[
\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \iff \quad \sum_{n=1}^{\infty} b_n \quad \text{converges},
\]

\[
\sum_{n=1}^{\infty} a_n \quad \text{diverges} \quad \iff \quad \sum_{n=1}^{\infty} b_n \quad \text{diverges}.
\]
To use this test given a series $\sum a_n$ we have to come up with a series $\sum b_n$ as our comparing series. Of course we must know the behavior of $\sum b_n$, but we can always default to the known p-series, either using $\sum \frac{1}{n}$ when we suspect divergence or $\sum \frac{1}{n^p}, p > 1$ when we suspect convergence. Here are examples of each case:

Example 5 Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ converges or diverges.

Solution

Here $a_n = \frac{n}{n^3 + 1}$ which we suspect converges because it has the form $\sum \frac{1}{n^2}$ for large $n$. thus, let

$b_n = \frac{1}{n^2}$

and compute

$r = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{\frac{n}{n^3 + 1}}{\frac{1}{n^2}} \right)$

Simplifying,
Since both series do the same thing. Since \( \sum \frac{1}{n^2} \) converges, so does \( \sum \frac{1 + n}{n^3 + 1} \).

**Example 6** Determine whether \( \sum_{n=1}^{\infty} \frac{2n^2 + 1}{5n^3 + 1} \) converges or diverges.

Solution

This time \( a_n = \frac{2n^2 + 1}{5n^3 + 1} \) behaves like \( \frac{1}{n} \) so we suspect divergence. Thus, we use \( b_n = \frac{1}{n} \) as our comparing sequence:

\[
r = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{2n^2 + 1}{5n^3 + 1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{2n^3 + n}{5n^3 + 1} = \frac{2}{5}
\]

Since \( 0 < r < 1 \) and \( \sum \frac{1}{n} \) diverges, so does \( \sum_{n=1}^{\infty} \frac{2n^2 + 1}{5n^3 + 1} \).

**Example 7** Determine whether \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \) converges or diverges.

Solution

Let us apply the limit comp test blindly; let \( b_n = \frac{1}{n} \) be our comparing sequence and let us see what limit we get:
This is case III of our result: If \( r = \infty \) and \( \sum b_n \) diverges, so does \( \sum a_n \).

Since \( r = \infty \) and \( \sum \frac{1}{n} \) diverges, so does \( \sum \frac{\ln(n)}{n} \).

**Example 8** Determine whether \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \) converges or diverges.

Solution

Once again, we blindly try \( b_n = \frac{1}{n} \):

\[
r = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{\ln(n)}{n^2}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n \cdot \ln(n)}{n^2} = 0
\]

You might be tempted to think that this is case II of our theorem but it is not; the limit is 0, but \( \sum b_n \) does not converge. Thus, nothing can be concluded; we should have used a different comparing sequence.

Let us try \( b_n = \frac{1}{n^2} \):

\[
r = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{\ln(n)}{n^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n \cdot \ln(n)}{n^2} = 0 \text{ (by l'Hopital's rule)}
\]
This is case II: If \( r = 0 \) and
\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]
converges, so does
\[
\sum_{n=1}^{\infty} \frac{\ln(n)}{n^a}.
\]

**Example 9** Determine whether \( \sum_{n=1}^{\infty} \frac{10^n}{(n + 1)!} \) converges or diverges.

**Solution**

This series involves the all important \( n! \) which we will encounter time and time again. Here,

\[
\alpha_n = \frac{10^n}{(n + 1)!}
\]

We should ask: What is \( \lim_{n \to \infty} \frac{10^n}{(n + 1)!} \) because this is what we should always do first. Clearly this is a case of \( \infty \to \infty \) but one in which we cannot use l’Hopital’s rule. How do we proceed? We use the definition of \( n! \):

\[
\frac{10^n}{(n + 1)!} = \frac{10 \times 10 \times 10 \times \cdots \times 10}{(n + 1)(n)(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1}
\]

The table below shows how the numerator and denominator behave:

<table>
<thead>
<tr>
<th>n-values</th>
<th>( 10^n )</th>
<th>( n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>10000</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>100000</td>
<td>120</td>
</tr>
<tr>
<td>6</td>
<td>1000000</td>
<td>720</td>
</tr>
</tbody>
</table>
You can see how much quicker $10^n$ grows and it is tempting to conclude
that $\lim_{n \to \infty} \frac{10^n}{(n + 1)!} = \infty$. However, look at the following table:

<table>
<thead>
<tr>
<th>n-Values</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>120</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>720</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>5040</td>
</tr>
</tbody>
</table>

It looks totally the opposite of the first table! This is why in math we cannot use
arguments based on observations; if we make an assertion, we must prove it and what
we want to prove here is that, in fact,

$$
\lim_{n \to \infty} \frac{10^n}{(n + 1)!} = 0
$$

Once we have done this, we will prove that $\sum_{n=1}^{\infty} \frac{10^n}{(n + 1)!}$ converges by either comparison test.

Exercise: show $\lim_{n \to \infty} \frac{10^n}{(n + 1)!} = 0$.

Let us use the comparison test by observing a simpler case:

$$
\frac{3^1}{(1 + 1)!} = \frac{3}{2}
$$

$$
\frac{3^2}{(2 + 1)!} = \frac{3 \times 3}{3 \times 2} = 1 \times \frac{3}{2}
$$

$$
\frac{3^3}{(3 + 1)!} = \frac{3 \times 3 \times 3}{4 \times 3 \times 2} = \frac{3}{4} \times 1 \times \frac{3}{2}
$$

$$
\frac{3^4}{(4 + 1)!} = \frac{3 \times 3 \times 3 \times 3}{5 \times 4 \times 3 \times 2} = \frac{3 \times 3}{4} \times \frac{3}{2} \leq \frac{3 \times 3}{4} \times \frac{3}{2}
$$

$$
\frac{3^5}{(5 + 1)!} = \frac{3 \times 3 \times 3 \times 3 \times 3}{6 \times 5 \times 4 \times 3 \times 2} = \frac{3 \times 3}{4} \times \frac{3}{2} \leq \frac{3 \times 3}{4} \times \frac{3}{2}
$$

By the time we get to the 5th term you can see that
\[
\frac{3^5}{(5 + 1)!} = \frac{3 \cdot (3^3)}{2 \cdot (4)}
\]

and you can verify that
\[
\frac{3^n}{(n + 1)!} = \frac{3 \cdot (3^{n-2})}{2 \cdot (4)} ; \quad n \geq 3
\]

Thus, \( \sum_{n=1}^{\infty} \frac{3^n}{(n + 1)!} \) converges since \( \sum_{n=1}^{\infty} \frac{3^n}{(4)} \) is a convergent geometric series and \( \sum_{n=1}^{\infty} \frac{3^n}{(n + 1)!} \) and \( \sum_{n=1}^{\infty} \frac{3^n}{(n + 1)!} \) differ only by a finite number of terms.

Now we reproduce the argument for \( \frac{10^n}{(n + 1)!} \). Once \( n \geq 10 \), we will have the following situation:

\[
\frac{10^n}{(n + 1)!} = \frac{10 \times 10 \cdots \times 10 \times 10}{(n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot 2} = \frac{10}{(n + 1)} \times \frac{10}{n} \times \frac{10}{(n - 1)} \times \cdots \times \frac{10}{2}
\]

Now we can argue in the same way as the previous case and write:

\[
\frac{10^n}{(n + 1)!} \leq \frac{10}{9} \times \frac{10}{11} \times \frac{10}{11} \times \cdots \times \frac{10}{2} \]

\[
\leq \left( \frac{10}{9} \times \frac{10}{11} \right)^{n-9}, \quad n \geq 10
\]

Thus, the comparison test tells us that \( \sum_{n=10}^{\infty} \frac{10^n}{(n + 1)!} \) converges because

\[
\sum_{n=10}^{\infty} \left( \frac{10}{9} \times \frac{10}{11} \right)^{n-9}
\]

is a convergent geometric series. Since the only difference
\[
\sum_{n=1}^{\infty} \frac{10^n}{(n + 1)!} \quad \text{and} \quad \sum_{n=10}^{\infty} \frac{10^n}{(n + 1)!}
\]

is a finite number of terms, \( \sum_{n=1}^{\infty} \frac{10^n}{(n + 1)!} \) also converges.